A small binomial theorem problem

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Question If $x + \frac{1}{x} = 1$, prove that $x^7 + \frac{1}{x^7} = 1$.

Solution

Method 1 $\left(x + \frac{1}{x}\right)^3 = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3} = \left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right)$

By given, $1^3 = \left(x^3 + \frac{1}{x^3}\right) + 3(1) \Rightarrow x^3 + \frac{1}{x^3} = 1 - 3 = -2$

$$\left(x + \frac{1}{x}\right)^5 = x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5} = \left(x^5 + \frac{1}{x^5}\right) + 5\left(x^3 + \frac{1}{x^3}\right) + 10\left(x + \frac{1}{x}\right)$$

$$\therefore \quad 1^5 = \left(x^5 + \frac{1}{x^5}\right) + 5(-2) + 10(1) \Rightarrow x^5 + \frac{1}{x^5} = 1$$

$$\left(x + \frac{1}{x}\right)^7 = \left(x^7 + \frac{1}{x^7}\right) + 7\left(x^5 + \frac{1}{x^5}\right) + 21\left(x^3 + \frac{1}{x^3}\right) + 35\left(x + \frac{1}{x}\right)$$

$$\therefore \quad (1)^7 = \left(x^7 + \frac{1}{x^7}\right) + 7(1) + 21(-2) + 35(1) \Rightarrow x^7 + \frac{1}{x^7} = 1$$

Method 2 $x + \frac{1}{x} = 1 \quad \Rightarrow x^2 + 1 = x \quad \Rightarrow x^2 - x + 1 = 0$

$$\Rightarrow x = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm \frac{\sqrt{-3}}{2} = a \pm b, \quad \text{where } a = \frac{1}{2}, b = \frac{\sqrt{-3}}{2}$$

Also, $\frac{1}{x} = \frac{2}{1 \pm \sqrt{-3}} = \frac{2\left(1 \mp \sqrt{-3}\right)}{\left(1 \pm \sqrt{-3}\right)\left(1 \mp \sqrt{-3}\right)} = \frac{2\left(1 \mp \sqrt{-3}\right)}{4} = \frac{1 \mp \sqrt{-3}}{2} = a \mp b$

$$x^7 + \frac{1}{x^7} = (a \pm b)^7 + (a \mp b)^7 = 2\left[a^7 + 21a^5b^2 + 35a^3b^4 + 7ab^6\right]$$

$$= 2\left[\left(\frac{1}{2}\right)^7 + 21\left(\frac{1}{2}\right)^5\left(\frac{-3}{2^2}\right) + 35\left(\frac{1}{2}\right)^3\left(\frac{9}{2^4}\right) + 7\left(\frac{1}{2}\right)\left(\frac{-27}{2^6}\right)\right] = 2\left[\frac{1 - 63 + 315 - 189}{2^7}\right] = 1$$

Method 3 $x + \frac{1}{x} = 1 \quad \Rightarrow x^2 + 1 = x \quad \Rightarrow x^2 - x + 1 = 0$

$$\Rightarrow x^r\left(x^2 - x + 1\right) = 0 \quad \Rightarrow x^{r+2} - x^{r+1} + x^r = 0 \qquad \text{.... } (*)$$

Now, $x^7 = \left(x^7 - x^6 + x^5\right) + \left(x^6 - x^5 + x^4\right) - \left(x^4 - x^3 + x^2\right) - \left(x^3 - x^2 + x\right) + x$

$= 0 + 0 - 0 - 0 + x$, by (*)

$= x$ ..... (1)

Divide (1) by $x^8 \neq 0$, we get $\frac{1}{x^7} = \frac{1}{x}$ .... (2)

(1) + (2), $x^7 + \frac{1}{x^7} = x + \frac{1}{x} = 1$.

Method 4

$$x+\frac{1}{x}=1 \qquad \Rightarrow x^2+1=x \quad \Rightarrow x^2-x+1=0 \qquad \ldots. \quad (1)$$

$$x^7+\frac{1}{x^7}-\left(x+\frac{1}{x}\right)=\left(x^7-x\right)-\left(\frac{1}{x}-\frac{1}{x^7}\right)=x\left(x^6-1\right)-\frac{1}{x^7}\left(x^6-1\right)=\left(x^6-1\right)\left(x-\frac{1}{x^7}\right)$$

$$=\left(x^3+1\right)\left(x^3-1\right)\left(x-\frac{1}{x^7}\right)=\left(x+1\right)\left(x^2-x+1\right)\left(x-1\right)\left(x^2+x+1\right)\left(x-\frac{1}{x^7}\right)=0 \quad \text{, by (1)}$$

$$\therefore \quad x^7+\frac{1}{x^7}=x+\frac{1}{x}=1$$

Method 5

$$x+\frac{1}{x}=1 \qquad \Rightarrow \left(x+\frac{1}{x}\right)^2=1^2 \qquad \Rightarrow x^2+2+\frac{1}{x^2}=1 \qquad \Rightarrow x^2+\frac{1}{x^2}=-1$$

$$x^3+\frac{1}{x^3}=\left(x^2+\frac{1}{x^2}\right)\left(x+\frac{1}{x}\right)-\left(x+\frac{1}{x}\right)=\left(-1\right)\left(1\right)-1=-2$$

$$x^5+\frac{1}{x^5}=\left(x^3+\frac{1}{x^3}\right)\left(x^2+\frac{1}{x^2}\right)-\left(x+\frac{1}{x}\right)=\left(-2\right)\left(-1\right)-1=1$$

$$x^7+\frac{1}{x^7}=\left(x^5+\frac{1}{x^5}\right)\left(x^2+\frac{1}{x^2}\right)-\left(x^3+\frac{1}{x^3}\right)=\left(1\right)\left(-1\right)-\left(-2\right)=1$$

Note: If $u_n=x^n+\frac{1}{x^n}$ then we have: $u_{n+k}=u_n u_k-u_{n-k}$, where $k<n$.

In particular, $u_{n+1}=u_n u_1-u_{n-1}$, where $n>1$.

Method 6 (For those who know complex number)

$$x+\frac{1}{x}=1 \qquad \Rightarrow x^2+1=x \quad \Rightarrow x^2-x+1=0$$

$$\Rightarrow x=\frac{1\pm\sqrt{-3}}{2}=\frac{1}{2}\pm i\frac{\sqrt{3}}{2}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$$

By de Morivres' Theorem,

$$x^7+\frac{1}{x^7}=\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^7+\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{-7}=\left(\cos\frac{7\pi}{3}+i\sin\frac{7\pi}{3}\right)+\left(\cos\left(-\frac{7\pi}{3}\right)+i\sin\left(-\frac{7\pi}{3}\right)\right)$$

$$=\left(\cos\frac{7\pi}{3}+i\sin\frac{7\pi}{3}\right)+\left(\cos\frac{7\pi}{3}-i\sin\frac{7\pi}{3}\right)=2\cos\frac{7\pi}{3}=2\cos\frac{\pi}{3}=2\times\frac{1}{2}=1$$